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Asymptotic structure of perturbative series for τ lepton observables

J.G. Körner[1], F. Krajewski[1], A.A. Pivovarov[1,2]

[1] Institut für Physik, Johannes-Gutenberg-Universität, Staudinger Weg 7, D-55099 Mainz, Germany
[2] Institute for Nuclear Research of the Russian Academy of Sciences, Moscow 117312, Russia

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Abstract. We analyze τ lepton decay observables, namely the moments of the hadronic spectral density in the finite energy interval $(0, M_\tau)$, within finite order perturbation theory including α_s^4 corrections. The start of the asymptotic growth of the perturbation theory series is found at this order in a scheme-invariant manner. We establish the ultimate accuracy of finite order perturbation theory predictions and discuss the construction of optimal observables.

1 Introduction

The study of τ lepton decays provides a wealth of information on low energy hadronic physics, where the accuracy of experimental data is permanently improving [1,2]. The central quantity of interest is the hadronic spectral density. The spectral density has been calculated with a very high degree of accuracy within perturbation theory (see, e.g., [3–5]). The structure of the observables – related to the two-point correlator of hadronic currents with well established and simple analytic properties – makes the comparison of experimental data with theoretical calculations very clean. All these features make τ lepton physics an important area of particle phenomenology where theory (QCD) can be confronted with experiment to a very high precision [6–10].

In the present note we show that within the finite order perturbation theory analysis the ultimate theoretical precision has been reached already now. The limit of precision exists due to the asymptotic nature of the perturbation theory series. The actual magnitude of this limiting precision depends on the numerical value of the coupling constant, which is the expansion parameter. We perform our analysis and reach our conclusions in a renormalization-scheme invariant way.

The normalized τ lepton decay rate into hadrons h is given by

$$R_\tau = \frac{\Gamma(\tau \to h\nu)}{\Gamma(\tau \to l\nu\bar{\nu})} = N_c(1+\delta), \tag{1}$$

with

$$R_\tau^{\rm exp} = 3.649 \pm 0.014 \quad \text{and} \quad \delta^{\rm exp} = 0.216 \pm 0.005. \tag{2}$$

The first term in (1) is the parton model result, while the second term δ represents the effects of the QCD interaction. In this paper we neglect electroweak corrections altogether. The theoretical expression for the rate is given by

$$R_\tau = N_c \int_0^{M_\tau^2} 2\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right)\rho(s)\frac{{\rm d}s}{M_\tau^2}. \tag{3}$$

The spectral density $\rho(s)$ is related to Adler's D function through the dispersion relation

$$D(Q^2) = Q^2 \int \frac{\rho(s){\rm d}s}{(s+Q^2)^2}. \tag{4}$$

The D function is computable in perturbation theory. In the $\overline{\rm MS}$ scheme the perturbation theory expression for the D function is given by

$$D(Q^2) = 1 + \frac{\alpha_s}{\pi} + 1.64\left(\frac{\alpha_s}{\pi}\right)^2 + 6.37\left(\frac{\alpha_s}{\pi}\right)^3 + k_3\left(\frac{\alpha_s}{\pi}\right)^4 + \ldots, \tag{5}$$

where the running coupling is normalized at the scale $\mu = Q$. The light quarks u, d and s are taken to be massless. Equations (3)–(5) constitute the full theoretical information necessary for the perturbation theory analysis of the τ system. The fourth order $\overline{\rm MS}$ scheme coefficient k_3 is not known at present.

In the present note we do not systematically discuss non-perturbative effects stemming from standard power corrections [11]. Also, the infinite resummation of the perturbation theory series different from the standard renormalization group improvement is used only as a toy example [12]. The standard power corrections due to nonvanishing vacuum expectation values of local operators within operator product expansion are relatively small and can be simply accounted for if desired. The coefficient functions of the local operators are known in low orders of the

perturbation theory expansions, and there is no necessity to thoroughly analyze their convergence properties. It is the high precision achieved in the experimental analysis of τ decays and the rather advanced stage of theoretical description that calls for a detailed analysis of the physics of the τ system.

2 Internal perturbation theory description of basic τ system observables

The central quantity of interest in the τ system is the hadronic spectral density which can be measured in the finite energy interval $(0, M_\tau = 1.777\,\mathrm{GeV})$. Being a distribution (in theory) or a rapidly varying function in the vicinity of resonances (in experiment) the hadronic spectral density cannot be analyzed pointwise within perturbation theory. The appropriate quantities to be analyzed are the moments (or generalized Fourier components over a chosen complete set of test functions). We define moments of the spectral density by (M_τ is chosen to be the unity of mass)

$$M_n = (n+1)\int_0^1 \rho(s) s^n \mathrm{d}s \equiv 1 + m_n. \tag{6}$$

Due to the completeness of the basis $\{s^n : n = 0, \ldots, \infty\}$ the moments m_n contain all information about $\rho(s)$. The invariant content of the investigation of the spectrum, i.e. independent of any definition of the charge, is the simultaneous analysis of all the moments. Note that within finite order perturbation theory the moments (6) coincide with the results of contour integration [13–16] because of the analytic properties of the functions $\ln^p s$.

In order to get rid of artificial scheme-dependent constants in the perturbation theory expressions for the moments we define an effective coupling $a(s)$ directly on the physical cut through the relation

$$\rho(s) = 1 + a(s). \tag{7}$$

All the constants that may appear due to a particular choice of the renormalization scheme are absorbed into the definition of the effective charge; see e.g. [17–20]. Note that if there were no running (as in the conformal limit of QCD with vanishing β function or at the infrared fixed point) then the whole physics of the τ system in the massless approximation (without strange particles, for instance, and including only perturbative corrections without possible power corrections) would reduce to the determination of a single number $a(M_\tau) \equiv a$, and consequently there would not be any problems with the convergence of the perturbation theory series. Because of the running of $a(s)$, however, different observables, i.e. different moments of the spectral density, generate different perturbation theory series from the original object $\rho(s)$ in (7). The whole set of moments needs to be analyzed in a scheme-invariant way [21–23]. Note that the introduction of a natural internal coupling parameter such as the effective charge $a(s)$, allows one to extend the perturbation theory series needed for the description of relations between observables by one more term as compared to the analysis in e.g. the $\overline{\mathrm{MS}}$ scheme [21,24]. When defining the effective charge directly through $\rho(s)$ itself we get theoretical perturbative corrections to the moments only because of the running. Without running one would have

$$M_n = 1 + a(M_\tau) \equiv 1 + a, \quad \text{or} \quad m_n \equiv a, \tag{8}$$

and the perturbation theory analysis would be over (we neglect power corrections for the moment!). In any given order of perturbation theory the running of the coupling $a(s)$ defined in (7) contains only logarithms of s with coefficients given by an effective β function:

$$\begin{aligned} a(s) = {} & a + \beta_0 L a^2 + (\beta_1 L + \beta_0^2 L^2) a^3 \\ & + \left(\beta_2 L + \frac{5}{2}\beta_1\beta_0 L^2 + \beta_0^3 L^3\right) a^4 + \ldots, \end{aligned} \tag{9}$$

where $a = a(M_\tau^2)$, and $L = \ln(M_\tau^2/s)$. The contributions of powers of logarithms to the normalized moments are

$$(n+1)\int_0^1 s^n \ln^p(1/s)\mathrm{d}s = \frac{p!}{(n+1)^p}. \tag{10}$$

Therefore, at fixed order of perturbation theory the effects of the running die out for large n improving the convergence of the perturbation theory series. With the definition of the charge according to (7) all high order corrections vanish at $n \to \infty$ at any fixed order of perturbation theory. With running one has instead of (8):

$$\begin{aligned} m_0 &= a + 2.25a^2 + 14.13a^3 + 87.66a^4 \\ &\quad + (433.3 + 4.5k_3)a^5, \\ m_1 &= a + 1.125a^2 + 4.531a^3 + 6.949a^4 \\ &\quad + (-175.2 + 2.25k_3)a^5, \\ m_2 &= a + 0.75a^2 + 2.458a^3 - 1.032a^4 \\ &\quad + (-142.6 + 1.5k_3)a^5, \\ m_3 &= a + 0.563a^2 + 1.633a^3 - 2.542a^4 \\ &\quad + (-110.4 + 1.125k_3)a^5, \\ &\quad \ldots, \\ m_{100} &= a + 0.022a^2 + 0.041a^3 - 0.25a^4 \\ &\quad + (-4.08 + 0.045k_3)a^5. \end{aligned} \tag{11}$$

For large n the moments behave better because the infrared region of integration is suppressed. Note that the coefficients of the series in (11) are saturated with the lowest power of logarithm for large n for a given order of perturbation theory, i.e. they are saturated with the highest coefficient of the effective β function.

Higher moments are not welcome from the experimental point of view. They are dominated by the contributions coming from the high energy end of the τ decay spectrum (therefore, they converge better perturbatively) but the experimental accuracy for the moments basically deteriorates with increasing n because the poorly known contributions close to the right end of the interval are enhanced.

To suppress experimental errors from the high energy end of the spectrum the modified system of mixed moments

$$\tilde{M}_{kl} = \frac{(k+l+1)!}{k!l!}\int_0^1 \rho(s)(1-s)^k s^l \mathrm{d}s \equiv 1 + \tilde{m}_{kl} \quad (12)$$

can be used [25]. The weight function $(1-s)^k s^l$ has its maximum at $\bar{s} = l/(l+k)$. The integral in (12) is dominated by contributions from around this value. A disadvantage of choosing such moments is that the $(1-s)^k$ factor enhances the infrared region strongly and ruins the perturbation theory convergence. As an example one has

$$\begin{aligned}
\tilde{m}_{00} &= a + 2.25a^2 + 14.13a^3 + 87.66a^4 \\
&\quad + (433.3 + 4.5k_3)a^5, \\
\tilde{m}_{10} &= a + 3.375a^2 + 23.72a^3 + 168.4a^4 \\
&\quad + (1042. + 6.75k_3)a^5, \\
\tilde{m}_{20} &= a + 4.125a^2 + 31.24a^3 + 241.1a^4 \\
&\quad + (1683. + 8.25k_3)a^5, \\
\tilde{m}_{30} &= a + 4.688a^2 + 37.51a^3 + 307.3a^4 \\
&\quad + (2324. + 9.375k_3)a^5. \qquad (13)
\end{aligned}$$

The values of the coefficients in the series (13) can be found in a concise form for arbitrary k at any giving finite order of perturbation theory. For instance, the contribution of the *log* term is given by

$$(k+1)\int_0^1 (1-s)^k \ln(1/s)\mathrm{d}s = \sum_{j=1}^{k+1}\frac{1}{j}. \quad (14)$$

In contrast to (10) it increases as $\ln(k)$ for large k, making the coefficients of the perturbation theory series large. The contribution of the $\log^2$ term reads

$$\begin{aligned}
&(k+1)\int_0^1 (1-s)^k \ln^2(1/s)\mathrm{d}s \\
&= \left(\sum_{j=1}^{k+1}\frac{1}{j}\right)^2 + \sum_{j=1}^{k+1}\frac{1}{j^2}, \qquad (15)
\end{aligned}$$

and can be seen to grow as $\ln^2(k)$ for large k.

In practical applications our formal criterion of the accuracy which the series provides is given by the numerical magnitude of the last term of the series. However, this criterion should be applied with great caution. Because of the freedom of the redefinition of the expansion parameter the last term of the series can always be made arbitrary small for any given observable. One can give an invariant meaning to the quality of the perturbation theory expansion only for a set of observables.

Before proceeding we would like to comment on the contribution of power corrections to the systems of moments (6) and (12) and the interplay between the magnitude of this contribution and the structure of the perturbation theory series. For the system of moments in (6), the contribution of power corrections reduces to a single term (neglecting the weak $log(Q)$ dependence of the coefficient functions of the operator product expansion which is a common practice) of the form $(\Lambda^2/M_\tau^2)^n$ which decreases very fast with n (Λ is a typical scale of power corrections related to the non-perturbative scale of QCD and $\Lambda < M_\tau$). This makes the perturbation theory contribution dominant in the total result. This perturbative term for the large n moments is saturated with high energy contributions and therefore converges perturbatively. The convergence becomes even better with increasing n. The moments are perturbatively dominated and, therefore, precise. On the contrary, for the system of mixed moments in (12) with $l \sim 0$ the large k moments are saturated with low energy contributions, i.e. basically with the contribution of the ground state resonance, and therefore are completely non-perturbative which is reflected in the fast deterioration of perturbative convergence. The contribution of power corrections to the moments (12) picks out many terms in all orders from $n = 2$ to $n = k+1$. Nothing definite can be said about such a sum of power corrections in any realistic case. This also indicates the importance of power corrections for mixed moments. The perturbation theory series is the same both for the vector and axial vector channels, while the lowest resonance contributions are completely different (the pion instead of the ρ meson). Therefore no method of summation of perturbation theory series can bring it to agree with experiment. In this case perturbation theory is in trouble and the power corrections provide the correct result for large k. Large k mixed moments, therefore, are not usable within the perturbation theory framework, even if they are preferable from the experimental point of view.

Thus, one faces the usual clash between experimental and theoretical accuracy which is reflected in our case in the range of (k,l) values for the mixed moments that are chosen as optimal observables. Having explicit perturbation theory formulas at hand, one can establish the ultimate theoretical accuracy implied by the asymptotic character of perturbation theory series for a given experimental observable with any stated precision. This allows one to conclude which error – experimental or theoretical – dominates the uncertainty of an observable related to τ decay physics.

For our numerical estimates we take $a = 0.111$ as obtained from the corresponding value of the $\overline{\mathrm{MS}}$ charge. From the set of moments $\{m_n; n = 0, \ldots,\}$ the moment m_0 has the largest infrared contribution. Therefore, a set of observables has the worst accuracy if the moment m_0 is included in the set. For m_0 one obtains

$$\begin{aligned}
m_0 &= 0.111 + 0.0277 + 0.0193 + 0.0133 \\
&\quad + (0.0073 + 0.000076k_3). \qquad (16)
\end{aligned}$$

As mentioned before the numerical value of the coefficient k_3 is unknown at present. In some of the following evaluations we want to fix its value to have a feeling of the importance of the last term of the perturbation theory series. One popular value is $k_3 = 25$ based on the Padé approximation. Another value, $k_3 = 91$, nullifies the fourth order coefficient of the effective β function [27]. Both these numbers are used only for illustrative purposes while our conclusion about perturbatively commensurate observables is

independent of k_3. Numerically for $k_3 = 25$ one has for the zeroth moment

$$m_0 = 0.111 + 0.0277 + 0.0193 + 0.0133 + \text{“}0.009''. \quad (17)$$

Formally, the convergence still persists even in (17) if one only requires subsequent terms of the series to decrease, but the convergence is very slow. Also, the total contribution of the four higher order terms is more than 60% of the leading one. For the first moment the convergence is considerably better:

$$m_1 = 0.111 + 0.014 + 0.006 + 0.001 + (-0.003 + 0.00004k_3), \quad (18)$$

and for $k_3 = 25$

$$m_1 = 0.111 + 0.014 + 0.006 + 0.001 - \text{“}0.002''. \quad (19)$$

With the choice $k_3 = 25$ the $O(a^5)$ term already shows numerical growth. If one keeps only the smallest term one gets a formal accuracy of about 1% and the total contribution of the three higher order terms gives only about 20% of the leading one. The large difference in accuracy between m_0 and m_1 is a general feature of the moment observables at fifth order of perturbation theory: one cannot get a uniform smallness at this order for several moments at the same time adjusting only one number, k_3. For $a = 0.111$ we therefore conclude that one is starting to see the onset of asymptotic growth at fifth order. The growth of the terms is independent of any definition of the charge if several moments are analyzed simultaneously and this feature cannot be changed by any choice of k_3. For any single moment, e.g. m_0, one can always redefine the charge and make the series converge well at any desired rate, but then other moments become bad in terms of this charge. The invariant statement about the asymptotic growth is that the system of moments m_n with $n = 0$ included cannot be treated perturbatively at the fifth order of perturbation theory for the numerical value of the expansion parameter $a = 0.111$ if one wants to obtain an accuracy better than 5–10%. This statement about the ultimate accuracy of the set of moment observables attainable in fifth order of perturbation theory is independent of whichever numerical value k_3 takes. If, however, the system of moments m_n does not include m_0 as an observable, a uniform accuracy better than 1% can be obtained for such a system within perturbation theory. For instance, excluding m_0 and using $k_3 \sim 100$ one can make the system of moments with $n \geq 1$ perturbation theory commensurate at fifth order in the sense that all fifth order terms can be made small simultaneously. To demonstrate this in a scheme-invariant way we choose the second moment (which is already well convergent) as a definition of our experimental charge and find

$$\begin{aligned} m_0 &= m_2 + 1.5m_2^2 + 9.417m_2^3 + 59.28m_2^4 + (310.3 + 3k_3)m_2^5, \\ m_1 &= m_2 + 0.375m_2^2 + 1.51m_2^3 + 2.527m_2^4 + (-54.45 + 0.75k_3)m_2^5, \\ m_2 &= m_2, \\ m_3 &= m_2 - 0.19m_2^2 - 0.544m_2^3 + 0.742m_2^4 + (35.2 - 0.375k_3)m_2^5, \\ m_4 &= m_2 - 0.3m_2^2 - 0.803m_2^3 + 1.69m_2^4 + (56.641 - 0.6k_3)m_2^5. \end{aligned} \quad (20)$$

The convergence for the moments m_1–m_4 (and for $n > 4$) is fine. The total contribution of higher order corrections is small. The worst series is the one for the zeroth order moment. Equation (20) shows that no choice of k_3 yields an accuracy for both m_0 and m_1 which is essentially better than the fourth order term. In fact, there is a narrow window, $40 < k_3 < 60$, where the formal criterion of convergence is satisfied for both m_0 and m_1, but we do not find it natural to rely on such a fine tuning; and even then the accuracy of the zero moment is only about 10%. This is an indication that the ultimate accuracy of the perturbation theory expansion for the zeroth moment has been reached. If the moment m_0 is excluded the choice $k_3 \sim 100$ allows one to make the convergence fast even to fifth order, and no conclusion about an asymptotic growth is possible.

The perturbation theory expansions for the system of moments with $(1 - s)^k$ weight shows worse behavior. With the above criterion of accuracy, the precision which is given by the series from (13) is of order 10–20% for the numerical value of a. This is not enough for a comparison with experiment at the present level of precision. For instance, an expansion of the higher moments in (13) in terms of the first one (which is the most perturbative one for this system) goes as follows:

$$\begin{aligned} \tilde{m}_{00} &\equiv m_0 = 0.17, \\ \tilde{m}_{10} &= 0.17 + 0.033 + 0.022 + 0.011 + (-0.005 + 0.00032k_3), \\ \tilde{m}_{20} &= 0.17 + 0.054 + 0.043 + 0.027 + (0.0015 + 0.00053k_3), \\ \tilde{m}_{30} &= 0.17 + 0.070 + 0.061 + 0.046 + (0.014 + 0.0007k_3). \end{aligned} \quad (21)$$

These series possess a formal accuracy of from 6% to 25%, and the contribution of higher order terms can be as large as the leading term. Because of the slow convergence there is no sign of improvement with higher orders of the perturbation theory: the series expansions do not allow any reliable estimate of the accuracy for large mixed moments. Also while for the moments (6) the total contribution of the corrections is small, the situation is different here. The total change of the leading order result due to higher order corrections is considerable and strongly differs for various moments. This is another indication that the set of mixed moments is not commensurate perturbatively.

3 τ decay rate

The τ decay width is given by a specific linear combination of moments. Because of the factor $(1 - s)^2$ present

in (3) the convergence property of the total decay rate observable is not optimal. The $(1-s)^2$ factor enhances the infrared region of integration, i.e. the relative magnitude of the contributions of logarithms $\ln(M_\tau^2/s)$ at small energy. The concrete shape of the weight function with the weight factor $(1-s)^2$ is the main source of the slow convergence. One has

$$r_\tau = a + 3.563a^2 + 24.97a^3 + 174.8a^4 + (1041. + 7.125k_3)a^5. \quad (22)$$

Using $a = 0.111$ and $k_3 = 25$,

$$r_\tau = 0.111 + 0.044 + 0.034 + 0.027 + \text{“}0.021''. \quad (23)$$

Formally, the consecutive terms decrease but the decrease is very slow. One can see that the pattern of convergence mainly follows that of the moment $\tilde{m}_{20}$ from (13) because of the factor $(1-s)^2$ in (3).

Equations (2) and (23) show the essence of the problem we are addressing. In the finite order perturbation theory analysis one has to compare $\delta^{\rm exp}$ with $\delta^{\rm th}$:

$$0.216 \pm 0.005 = \delta^{\rm exp}, \quad \text{versus}$$
$$\delta^{\rm th} = 0.111 + 0.044 + 0.034 + 0.027 + \text{“}0.021'', \quad (24)$$

and the uncertainty of the theoretical expression “0.021” (or even 0.027) is much larger than the experimental error of 0.005. Thus, the theoretical uncertainty due to the truncation of the perturbation theory series is much larger than the experimental error of the corresponding observable. The common practical expectation for theoretical perturbation theory expansions to be useful is the smallness of the total higher order corrections, if nothing is known about the convergence of the expansion. For the rate observable the corrections increase the leading order result by a factor of 2. Note that one can improve the explicit convergence of the rate observable by a special redefinition of the expansion parameter due to renormalization scheme freedom. However, then the first moment of the differential decay rate will behave wildly. It is this feature that prompts us to reach definite conclusions about the asymptotic growth of perturbation theory expansion independent of any scheme. Two different sets of observables, where one set includes the moment m_0 and the other set does not include it, are not perturbatively connected with an accuracy required by experiment. Indeed, the first s moment of the differential decay rate $\mathrm{d}R_\tau/\mathrm{d}s$ gives the series with a faster convergence than (22):

$$r_\tau^{(1)} = a + 2.138a^2 + 10.15a^3 + 28.43a^4 + (-268.3 + 4.275k_3)a^5, \quad (25)$$

or numerically with $k_3 = 25$:

$$r_\tau^{(1)} = 0.111 + 0.026 + 0.014 + 0.004 - \text{“}0.003''. \quad (26)$$

The second s moment has an even better perturbative expansion:

$$r_\tau^{(2)} = a + 1.575a^2 + 6.186a^3 + 6.386a^4 + (-283.3 + 3.15k_3)a^5, \quad (27)$$

and numerically with $k_3 = 25$:

$$r_\tau^{(2)} = 0.111 + 0.0194 + 0.0085 + 0.001 - \text{“}0.003''. \quad (28)$$

The fifth order term is larger than the fourth order term for $k_3 = 25$. No choice of k_3 can simultaneously make all these three observables convergent at fifth order. If one chooses $k_3 \sim 100$ in order to guarantee a better convergence of the higher moments (which is physically motivated) one almost destroys the perturbation theory series for the decay rate (22).

The $(1-s)^n$ moments of the differential decay rate suppress poorly known high energy experimental data. Taking $n = 1$ as an example one has

$$r_\tau^{(1-s)} = a + 4.173a^2 + 31.31a^3 + 237.6a^4 + (1603. + 8.35k_3)a^5, \quad (29)$$

and numerically for $k_3 = 25$

$$r_\tau^{(1-s)} = 0.111 + 0.051 + 0.043 + 0.036 + \text{“}0.031''. \quad (30)$$

For $k_3 = 100$ the series reads

$$r_\tau^{(1-s)} = 0.111 + 0.051 + 0.043 + 0.036 + \text{“}0.041'', \quad (31)$$

which gives only about 30% accuracy and more than a factor 2 for the change of the leading order term. We conclude that the theoretical precision cannot compete with the experimental precision.

There are two distinct problems in analyzing τ decays: one is to describe the set of observables of the system using its internal coupling parameter defined to get the highest precision, and establish whether the set is perturbatively commensurate, while another is to extract the standard $\overline{\rm MS}$ parameters. It can happen that the set of observables is perturbatively connected with some given accuracy but the $\overline{\rm MS}$ coupling $\alpha_{\rm s}$ is not the best parameter for the expansion. This is the case here. In internal terms the τ system is described with higher accuracy in terms of the number of perturbation theory terms than in the $\overline{\rm MS}$ scheme. However, at this level of expansion one sees the asymptotic growth of the perturbation series for the numerical value of the expansion parameter fixed by experiment.

The expression for the decay rate in the $\overline{\rm MS}$ scheme possesses only $O(\alpha_{\rm s}^3)$ accuracy:

$$r_\tau = \left(\frac{\alpha_{\rm s}}{\pi}\right) + 5.20\left(\frac{\alpha_{\rm s}}{\pi}\right)^2 + 26.4\left(\frac{\alpha_{\rm s}}{\pi}\right)^3 + (78.0 + k_3)\left(\frac{\alpha_{\rm s}}{\pi}\right)^4, \quad (32)$$

with a numerical precision of only 30% again. A numerical value for $\alpha_{\rm s}$ is usually extracted treating the three first terms of the rate expression (32) as an exact function. The numerical value found is rather precise. However, the accuracy of the numerical prediction for other observables is dominated by the uncertainty of the truncation of the series and is poor if the observable contains the zeroth order moment. Therefore, the comparison of

different observables of the system cannot be done with high precision, and the ultimate precision is limited by the asymptotic growth of the perturbation theory series. The coupling constant, though important, is still an artificial parameter and the knowledge of its precise numerical value does not suffice for computing observables with sufficiently high precision.

The investigation of the τ system can be performed in N^{4}LO without any free parameters with the use of the internal charge a (and even N^{5}LO with the single free parameter k_3, which does not affect the conclusion about the asymptotic structure of perturbation theory series). However, the $\overline{\rm MS}$ scheme coupling can be expressed through a only up to NNLO because of the unknown coefficient k_3. The extraction of the $\overline{\rm MS}$ charge from a can be made through the relation

$$\frac{\alpha_{\rm s}^{\overline{\rm MS}}(M_\tau)}{\pi} = a - 1.64a^2 + 15.7a^3 + (49.6 - k_3)a^4 + \ldots, \tag{33}$$

with a reasonably fast convergence for $k_3 = 25$ or $k_3 = 100$.

4 Infrared fixed point as a model for an infinite perturbation theory series

The accuracy of approximating a function with the sum of a finite number of term of its asymptotic series strongly depends on the analytic structure of this function. Generally, there is an infinite number of ways to sum an asymptotic series with quite different results. Therefore, estimates of the accuracy based on the asymptotic series alone can be rather misleading. To discuss this issue in more detail (though for illustrative purposes only), we consider a model for the exact function as a source for the perturbative expansion (or recipe of the infinite resummation). The model uses the existence of the infrared fixed point for the running coupling in the third order of perturbation theory which allows one to extrapolate running to the origin. In this particular case we can compare perturbative expansions with an exact answer. This example allows one to check the general conclusions about the asymptotic structure and the divergence of the series, even if this is done in a model-dependent way. The effective β function is given by

$$\beta_{\rm eff}(a) = -\frac{9}{4}a^2 - 4a^3 + 25.7a^4 + \left(409.5 - \frac{9}{2}k_3\right)a^5 + O(a^6), \tag{34}$$

where the only free parameter is k_3 because the β_3 coefficient in the $\overline{\rm MS}$ scheme is known [26]. The third order approximation of the β function (34) possesses an infrared fixed point with the value $a_f = a(0) = 0.384$.

The integration in (6) can be explicitly performed with the third order β function from (34). With the initial value $a = 0.111$ one obtains $m_0^{\rm f} = 0.1605$, $m_1^{\rm f} = 0.130954$ to be compared with the results of (11), (16 and (18). The naive estimate of the accuracy does not always work for this resummation recipe for all the moments. Indeed, having the explicit model at hand one can generate an arbitrary number of terms of the perturbative expansion. For the zeroth moment the series diverges with the pattern

$$m_0^{\rm fix} = 0.111 + 0.028 + 0.019 + 0.013 + 0.014 + 0.018 + 0.029 + 0.053 + 0.114 + \ldots, \tag{35}$$

giving an ultimate accuracy of only about 10%. This accuracy is obtained keeping the smallest term. The sum of the first four terms gives the best estimate,

$$m_0^{\rm fix,best} = 0.111 + 0.028 + 0.019 + 0.013 = 0.171 \pm 0.013,$$

to be compared with the exact result $m_0^{\rm f} = 0.1605$. The central value is a bit too high but still within the uncertainty given by the last term.

For $m_1^{\rm fix}$ one also finds a divergent series but with a much faster decrease of the first few terms. The pattern of "convergence" is given by the following huge expression which we display to show how complicated things can become. One has

$$\begin{aligned} m_1^{\rm fix} = {} & 0.111 + 0.013861 + 0.006197 \\ & + 0.001054 + 0.000480 + 0.000088 \\ & + 0.000053 + 0.000016 + 0.000015 \\ & + 0.000014 + 0.000019 + 0.000026 \\ & + 0.000042 + 0.000072 + 0.000135 \\ & + 0.000268 + 0.000568 + 0.001277 \\ & + 0.003 + 0.0076 + 0.01997 + 0.055 + \ldots \end{aligned} \tag{36}$$

The best estimate is formally given by the sum of the first ten terms

$$m_1^{\rm fix,best} = 0.132795 \pm 0.000014, \tag{37}$$

according to the formal prescription for the evaluation of precision. The exact result $m_1^{\rm f} = 0.130954$, however, does not fall into the tiny interval given by the error bars in (37). Therefore, the formal criterion of the accuracy is violated in this case: the discrepancy $m_1^{\rm fix,best} - m_1^{\rm f} = 0.00184$ is not controlled by the smallest term of the asymptotic expansion (36). Still this discrepancy is small and the actual accuracy for the first moment given by the asymptotic expansion (36) is 1.3%. This suffices for practical purposes. One can easily see the difference between these two observables which reflects the different numerical magnitudes of the infrared contributions. Note also that the numerical magnitude of the smallest term of the expansion (36) is very sensitive to the value of the third coefficient of the effective β function. For the decay rate we find $r_\tau^{\rm f} = 0.1946$ and $r_\tau^{\rm f}(1) = 0.1527$ to be compared with (22, (23 and (24), and (25) and (26).

From (34) one sees that an infrared fixed point exists also in fourth order of the effective β function for any

$k_3 < 95.9$. For these numerical values the effect of k_3 on the exact moments within the model is rather weak. The pattern of "convergence" for the decay rate is mainly determined by the contribution of the zeroth order moment (or even by the mixed $\tilde{m}_{20}$ moment) and is very close to the expressions in (13)–(35).

In this model there are ways of accelerating the convergence with nice results but they definitely cannot be justified for use in the general case. Still our conclusion about the achievable precision within finite order perturbation theory in fifth order remains valid.

5 Conclusions

Using the standard estimate of the accuracy of an asymptotic series we have found that the theoretical precision in the perturbative description of τ decay is already limited by the asymptotic growth of the coefficients in fifth order of perturbation theory. This is a scheme-invariant statement. The accuracy of perturbative expansions for a reasonably general set of observables cannot be better than 5–10%. Taking a stricter attitude we claim that the zeroth order moment is not computable within perturbation theory. Any consistent description of τ decay data at fifth order of perturbation theory requires the exclusion of the zeroth order moment from the list of observables (or it should not constitute a dominant contribution). At fifth order of perturbation theory and with the present numerical value of the coupling, the first two moments of the spectral density are too different to be simultaneously treated by perturbation theory with an accuracy better than 5–10%. Therefore, one has to go beyond finite order perturbation theory to compare these two observables if one requires a theoretical accuracy that exceeds the present experimental accuracy. This implies the use of some procedure of resummation. The resummation procedure is not defined uniquely and the result depends on the prescription chosen [16,24,27]. Moreover, if one resums the infinite number of perturbation theory terms the condensates have no invariant meaning anymore, and their numerical values may change [24]. Therefore, improving the theoretical accuracy for this system seems to require the creation of a new paradigm.

The extraction of α_s from the τ decay rate and its comparison with the α_s values determined from other experiments does not appear to be the best test of perturbation theory for the τ system. The crucial test of the applicability of perturbation theory for the τ system would be the simultaneous calculation of two observables (moments) with an appropriate accuracy. If the set of moments includes the zeroth moment then the ultimate accuracy of finite order perturbative expansions has already been reached.

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